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Addendum to "A Third-Order Optimum Property of the Maximum Likelihood Estimator"

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A third-order optimum property of the maximum likelihood estimator is extended to not necessarily symmetric loss functions under an appropriate restriction on the class of competing estimators.

This note is a corollary to Pfanzagl and Wefelmeyer [5]. The terminology and notations of this paper will be used without further comments.

In [5, Theorem 1, p. 6] it was shown that the class of appropriately corrected m.l. estimator-sequences is as. complete of third order (i.e. up to an error term $o(n^{-1})$) within the class of estimator-sequences admitting stochastic expansions, provided the estimators are evaluated by the risks with respect to a symmetric unimodal loss function.

In this note it is shown that this class is even as. complete for arbitrary (i.e. not necessarily symmetric) unimodal loss functions, if the comparison is restricted to estimator-sequences admitting a stochastic expansion of type (3.1) with $(u, v) \to Q_1(u, v, \theta)$ symmetric and $(u, v, w) \to Q_2(u, v, w, \theta)$ antisymmetric. The class of these estimator-sequences will be denoted by $\mathscr{T}$.

The interest in such special estimator-sequences is motivated by the fact that the estimator-sequences obtained by the common methods are of this type. For the m.l. estimator see [5, Lemma 5.11, p. 17]; for Bayes and maximum posterior density estimators see Gusev [4, Theorem 5, p. 489, resp. Theorem 1, p. 476]; for the median of the posterior density see Strasser [6, Theorem 4, p. 32]. This is due to the fact that these estimators are defined as zeros of certain functions of the parameter. The Taylor expansion of such a function about the true parameter leads to a polynomial in the difference between the estimator and the true parameter. The inversion of this polynomial leads to a stochastic expansion of the estimator-sequence with Q_1 containing powers 0 and 2, Q_2 containing powers 1 and 3 only.

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The Result

Let $\mathscr{L}$ denote the class of all measurable functions $L: \mathbb{R}^p \to \mathbb{R}$ which are nonnegative, bounded and neg-unimodal about zero. As in Pfanzagl and Wefelmeyer [5] let $\mathscr{Q}$ denote the class of functions $q: \mathbb{R}^p \to \mathbb{R}^p$ the components of which admit partial derivatives fulfilling local Lipschitz conditions.

We obtain the following modifications of [5, Theorem 1, p. 6, Corollary 1, p. 9].

THEOREM. *Let the assumptions of Theorem* 1 *be fulfilled. Let* $\theta^{(n)}$, $n \in \mathbb{N}$, *be an estimator-sequence which is asymptotically maximum likelihood of order* $n^{-3/2}l_n(2)$. *Then the following assertions hold.*

(a) *For every estimator-sequence* $T^{(n)}$, $n \in \mathbb{N}$, *in* $\mathscr{T}$ *there exists a function* $q \in \mathscr{Q}$ *such that for every* $L \in \mathscr{L}$ *we have locally uniformly in* θ,

$$P_\theta{}^n(L(n^{1/2}(\theta^{(n)} + n^{-1}q(\theta^{(n)}) - \theta))) \leqslant P_\theta{}^n(L(n^{1/2}(T^{(n)} - \theta))) + o(n^{-1}).$$

(b) *Let* $L_* \in \mathscr{L}$ *be a loss function admitting bounded partial derivatives a.e. with respect to Lebesgue measure, such that*

$$\left(\int \varphi_{\Lambda(\theta)}(u)\, L_*^{\tau}(u) u_\beta\, du\right)_{\tau,\beta=1,\ldots,p}$$

is nonsingular for all $\theta \in \Theta$.

Then there exists $q^* \in \mathscr{Q}$ *such that* $\theta^{(n)} + n^{-1}\, q^*(\theta^{(n)})$ *is* L_*-*unbiased up to* $o(n^{-1/2})$. *If* $T^{(n)}$, $n \in \mathbb{N}$, *is any other estimator-sequence in* $\mathscr{T}$ *which is* L_*-*unbiased up to* $o(n^{-1/2})$, *then for every* $L \in \mathscr{L}$ *we have locally uniformly in* θ,

$$P_\theta{}^n(L(n^{1/2}(\theta^{(n)} + n^{-1}q^*(\theta^{(n)}) - \theta))) \leqslant P_\theta{}^n(L(n^{1/2}(T^{(n)} - \theta))) + o(n^{-1}).$$

Remark 1. It is easily seen that, as in Theorem 1, the Cramér-type conditions required here can be replaced by smoothness conditions on the loss functions, if the result of Götze and Hipp [2, Theorem (3.6), p. 71] is employed.

Remark 2. J. K. Ghosh, B. K. Sinha, and H. S. Wieand [3] consider, for the one-dimensional case, estimator-sequences the distributions of which admit Edgeworth-expansions. They identify a certain type of expansion for which a result analogous to the Theorem holds. In other words: They use an assumption on the Edgeworth-expansion instead of the symmetry—anti-symmetry condition on the stochastic expansion used here.

Remark 3. The Theorem applies in particular to curved exponential families if only as. efficient locally stable estimator-sequences are considered which are corrected to achieve as. unbiasedness. (It can be easily checked that the stochastic expansions of such estimators fulfill the symmetry-antisymmetry condition.)

For the one-dimensional case, J. K. Ghosh, B. K. Sinha, and H. S. Wieand [3] obtain this result by a computation of the cumulants. The same idea is employed by Takeuchi and Akahira [1, Theorem 6.2, p. 293] in the multidimensional case.

We mention this fact for the sake of completeness, despite our inability to see any operational justification for restricting the competing estimator-sequences to locally stable ones.

Proof of the Theorem. By [5, Lemma 11, p. 17] $\theta^{(n)}$, $n \in \mathbb{N}$, is in $\mathscr{T}$, and the same holds for $\theta^{(n)} + n^{-1}q(\theta^{(n)})$, $n \in \mathbb{N}$. Consider the Lebesgue-density [5, (6.18)] of the Edgeworth-expansion of the distribution of an estimator-sequence in $\mathscr{T}$. The $n^{-1/2}$-term of the risk depends on the estimator only through q. In [5, proof of Theorem 1] the symmetry of the loss function was used to show that, in computing the n^{-1}-term of the risk by integration over u, the terms containing odd powers of u vanish. If Q_1 is symmetric and Q_2 is antisymmetric, these terms vanish already in [5, (6.18)], since the functions $F_{\alpha\beta}$ and $E_{\alpha\beta\gamma\delta}$ are antisymmetric. (This has to be checked by computing these terms explicitly.) Hence the n^{-1}-term of the risk is the same as the one given in [5, p. 26], if the symmetry of L is replaced by the symmetry-antisymmetry condition on the stochastic expansion; and the proof can be concluded in the same way, since [5, Lemma 5.8] holds for arbitrary functions f which are negunimodal about zero (not only for symmetric ones).

This more general version of [5, Lemma 5.8] follows easily from the fact that [5, Corollary 5.7] holds if "symmetric about zero and unimodal [resp. star up]" is replaced by "unimodal [resp. star up] about zero". Then the function h_0 occurring in [5, proof of Lemma 5.8] is star up about zero. In the lines 16_7 to 16_5 replace $h_0(y)$ by $(h_0(y) + h_0(-y))/2$ and use

$$\int \varphi_{\sigma^2}(y)(y^2 - \sigma^2)\, h_0(y)\, dy = \int \varphi_{\sigma^2}(y)(y^2 - \sigma^2)\, h_0(-y)\, dy.$$

Correction. Part (a) of Condition B [5, p. 14] should be replaced by the following Lipschitz condition:

For every $\theta \in \Theta$ there exist constants $a, b > 0$ such that for $r, r' \in \mathbb{R}^k$ with $\| r' \| \leqslant \| r \|$

$$| Q(r, \theta) - Q(r', \theta)| \leqslant \| r - r' \| (a + \| r \|^b).$$

References

[1] Takeuchi, K. and Akahira, M., (1978). Third order asymptotic efficiency of maximum likelihood estimator for multiparameter exponential case. *Rep. Univ. Electro-Commun.* 28 271–293.

[2] GÖTZE, F., AND HIPP, C. (1978). Asymptotic expansions in the central limit theorem under moment conditions. *Z. Wahrscheinlichkeitstheorie Verw. Gebiete* **42** 67–87.
[3] GHOSH, J. K., SINHA, B. K., AND WIEAND, H. S. (1977). Second order efficiency of the MLE with respect to any bounded bowl-shaped loss function, to appear.
[4] GUSEV, S. I. (1975). Asymptotic expansions associated with some statistical estimators in the smooth case. I. Expansions of random variables. *Theor. Probability Appl.* **20** 470–498.
[5] PFANZAGL, J., AND WEFELMEYER, W. (1978). A third-order optimum property of the maximum likelihood estimator. *J. Multivar. Anal.* **8** 1–29.
[6] STRASSER, H. (1977). Asymptotic expansions for Bayes procedures. In *Recent Developments in Statistics* (J. R. Barra *et al.*, Eds.), pp. 9–35. North–Holland, Amsterdam.

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